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NUMERICAL MODELING OF A PLANE HYDRODYNAMIC PROBLEM FOR A VISCOELASTIC LUBRICANT. THE CASE OF SHORT RELAXATION TIMES

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Numerical solutions of a two-dimensional isothermal problem of flow of a relaxing fluid between two rotating cylindrical surfaces have been constructed. All the results have been obtained by solution of a complete system of hydrodynamic equations. The influence of the internal relaxation time on the carrying capacity of a lubricating film and the flow structure has been analyzed.

The classical theory of lubrication is based on a number of propositions widely used in modeling of frictional units [1–3]:

(1) the rheology of a lubricating material is described by the Navier–Stokes model;

(2) reduction of the hydrodynamic equations to a simplified Reynolds system enabling one to determine the functional form of the dependence of unknown quantities on one spatial coordinate is possible.

At the same time, it is well known that there can be situations where a particle of a lubricating material traverses the region of large pressure gradients over a period comparable to the time of relaxation to a local thermodynamic equilibrium. In this case, one must use viscoelastic rheological models allowing for the internal relaxation processes in the lubricant.

We have proposed in [4] a generalization of the Reynolds theory for the steady-state regime of plane viscoelastic-lubricant flow without a slip with two additional conditions: (a) the characteristic relaxation time τ_1 is longer than the characteristic time of traversal of the region of large gradients by a particle τ_0; (b) the flow velocity everywhere differs only slightly from a fixed value determined by the motion of solid surfaces that bound the region of flow. In this case, an integrodifferential equation containing the relaxation kernel is obtained instead of the ordinary differential equation for pressure. This integrodifferential equation is solved by numerical methods [4]. If the above conditions are not fulfilled, the problem cannot be reduced to a one-dimensional equation.

There is no total clarity in description of the behavior of solutions in the region of high values of the ratio $s = \tau_0/\tau_1$. It is clear a priori that when s tends to infinity, transition to a classical Reynolds theory must asymptotically be carried out [1–3]. However, the behavior of solutions in the transition region and the range of s corresponding to this transition region are unknown. In the present work, we consider a two-dimensional hydrodynamic problem for a viscoelastic lubricant in the absence of slip. The fulfillment of conditions (a) and (b) is not assumed and reduction of the problem to a one-dimensional equation is impossible. A number of solutions on the basis of a system of hydrodynamic equations have been obtained by direct numerical modeling for different values of the parameter s.

The geometry of the region of flow will be taken to be analogous to the problem on alignment of a shaft (Fig. 1), which has been posed by R. A. Turusov and has been considered in [2] within the framework of the approximation yielded by the Reynolds theory. The shaft (solid cylinder of radius R_{sh}) is placed within a ring (hollow cylinder of radius R_r). The entire space between the shaft and the ring is filled with a compressible viscoelastic fluid (lubricant). The distance between the centers of the shaft and the ring (eccentricity) is ε = ShR. Both the shaft and the ring are set in clockwise running (rotation) with the same linear velocities, i.e., $\omega_{sh}R_{sh} = \omega_r R_r$. It is necessary to find the steady-state isothermal flow of the fluid in the gap. Flow along the third axis, which is in parallel to the shaft axis, is taken to be insignificant.

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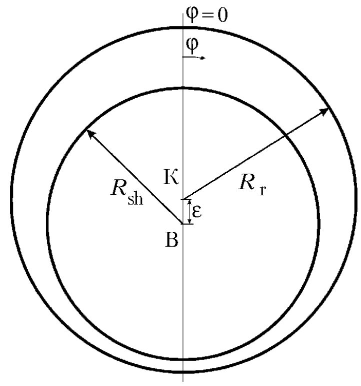


Fig. 1. Geometry of the region of flow.

To formulate governing equations it is convenient to use the two-dimensional curvilinear coordinate system y^α, $\alpha = 1, 2$, geometrically connected with the region of flow. We set the coordinate y^1 equal to the angle φ with the origin at the upper point (see Fig. 1). The second dimensionless coordinate $0 \le y^2 \le 1$ characterizes the position of a point of the flow region between the points (lying on one radius) of the shaft and the ring in accordance with the equation $h(\varphi) = y^2 h_{sh}(\varphi) + (1 - y^2) h_r(\varphi)$, where $h(\varphi)$ is the transverse coordinate between the shaft and the ring; this coordinate corresponds to the angle φ. The geometric parameters of the channel will be thought of as being related by the approximate relation $h_r(\varphi) - h_{sh}(\varphi) = \Delta\,(1 - \lambda \cos(\varphi - \pi))$, where $\Delta = R_r - R_{sh}$ and $\lambda = \varepsilon \Delta^{-1}$ [2].

A complete system of hydrodynamic equations that describes the motion of the fluid in the gap between the shaft and the ring involves one mass equation and two momentum equations [5]. In projection onto the axes of the Cartesian coordinate system x^a, $a = 1, 2$, with the use of the tensor notation [6], this system of equations can be written in a conservative form:

$$\left(g^{1/2} \rho v^\alpha b^a_\alpha\right)_{,t} + \left(g^{1/2} \pi^{\alpha\beta} b^a_\alpha\right)_{,\beta} = 0\,, \tag{1}$$

$$\left(g^{1/2} \rho\right)_{,t} + \left(g^{1/2} \rho v^\alpha\right)_{,\alpha} = 0\,. \tag{2}$$

Here and below, we carry out summation over double spatial indices; the Greek indices correspond to the curvilinear coordinates, whereas the Latin indices correspond to the Cartesian coordinates:

$$\pi^{\alpha\beta} = \rho v^\alpha v^\beta + p g^{\alpha\beta} - \tau^{\alpha\beta}; \quad g_{\alpha\beta} = \frac{\partial x^a}{\partial y^\alpha} \frac{\partial x^a}{\partial y^\beta}\,; \quad b^a_\alpha = \frac{\partial x^a}{\partial y^\alpha}\,, \quad g = \det(g_{\alpha\beta})\,.$$

The metric tensor can be used in the regular manner for lowering and raising of the Greek indices [6]. We take as the equation of state the approximation of a weakly compressible fluid

$$p = E\,(\rho - \rho_*)\,\rho_*^{-1} + p_*\,. \tag{3}$$

In numerical calculations, we took the following specific values: $\rho_* = 894.3$ kg/m^3 and $E = 1.02 \cdot 10^9$ Pa.

To formulate the rheological law for the lubricant, i.e., the relationship between the viscous stress tensor $\tau^{\alpha\beta}$ and the strain-rate tensor $e^{\alpha\beta} = \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} (v_{\gamma,\delta} + v_{\delta,\gamma})$ it is convenient to separate the spherical and deviation components of these tensors:

$$\tau_s = \tau^{\alpha\beta} g_{\alpha\beta}\,, \quad e_s = e^{\alpha\beta} g_{\alpha\beta}\,,$$

$$\tau_{\rm d}^{ab} = b_{\alpha}^{a} b_{\beta}^{b} \left(\tau^{\alpha\beta} - \frac{1}{3} g^{\alpha\beta} \tau_{\rm s} \right), \quad e_{\rm d}^{ab} = b_{\alpha}^{a} b_{\beta}^{b} \left(e^{\alpha\beta} - \frac{1}{3} g^{\alpha\beta} e_{\rm s} \right).$$

We take the following rheological relationships:

$$\tau_{\rm s} = \eta e_{\rm s} , \tag{4}$$

$$\tau_{\rm d}^{ab} (t) = 2\mu \int K (t - t_0) \, e_{\rm d}^{ab} (t_0) \, dt_0 . \tag{5}$$

The domain of time integration in (5) is a particle of a medium (not a spatial point!). The relaxation kernel $K(t)$ is assumed to be independent of pressure. It is convenient to normalize the kernel by the condition

$$\int K (t) \, dt = 1 . \tag{6}$$

In this case, the quantities η and μ can be interpreted as the ordinary coefficients of volume and shear viscosity, observed in slow flows of this fluid.

The functional form of the relaxation kernel was discussed earlier in [4, 7–10]. Putting it briefly, it is necessary to fulfill the following conditions:

(1) $K(t) = 0$ at $t < 0$ (causality condition);

(2) $\mathrm{Re} \int \exp (i\omega t) K(t) dt \geq 0$ (condition of consistency of the model with the second law of thermodynamics);

(3) $0 < K(0) < +\infty$ (condition of finiteness of the velocity of the signal).

Experimental investigations [11] performed for a number of fluids point to the functional form of the kernel with two discrete points τ_1 and τ_2 in the spectrum of internal relaxation times and with a continuous portion $0 < \tau < \tau_3$; here, we have $\tau_1 > \tau_3$ and $\tau_2 > \tau_3$:

$$K (t) = A_1 \tau_1^{-1} \exp (- t/\tau_1) + A_2 \tau_2^{-1} \exp (- t/\tau_2) + \int_0^{\tau_3} A (\tau) \, \tau^{-1} \exp (- t/\tau) \, d\tau , \quad t \geq 0 , \tag{7}$$

where the weight factors are bounded by the additional relations

$$A_1, A_2, A (\tau) \geq 0 , \quad A_1 + A_2 + \int_0^{\tau_3} A (\tau) \, d\tau = 1 .$$

It is easy to check that the relaxation kernel (7) satisfies the enumerated conditions (1), (2), and (3) above and also (6). In the present work, we have taken the simplest partial form of kernel (7):

$$K (t) = \tau_1^{-1} \exp (- t/\tau_1) . \tag{8}$$

Let us discuss the boundary conditions. For the velocity vector we specified:

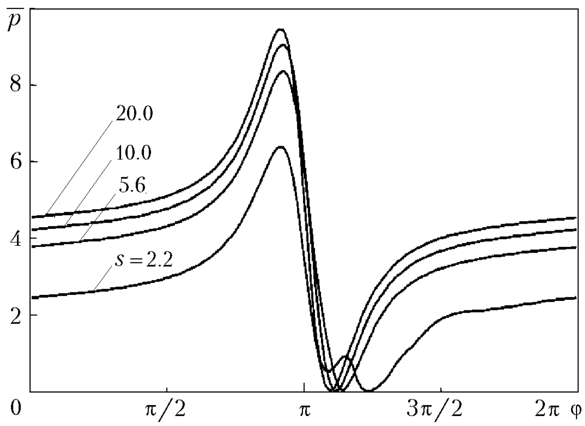
(a) the adhesion conditions, which, with allowance for the rotation of the shaft and the ring with linear velocity U, have, in this case, the form

$$v^1 (y^1, 0) = U/h_{\rm r} (\varphi) , \quad v^1 (y^1, 1) = U/h_{\rm sh} (\varphi) ; \tag{9}$$

(b) the conditions of nonflow through solid surfaces

$$v^2 (y^1, 0) = 0 , \quad v^2 (y^1, 1) = 0 ; \tag{10}$$

(c) the periodic boundary conditions along the coordinate y^1



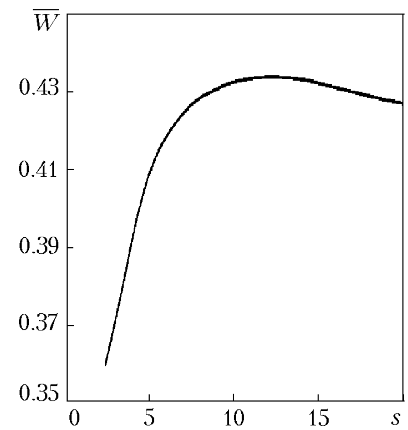


Fig. 2. Distribution of the dimensionless pressure on the shaft surface.

Fig. 3. Carrying capacity of the lubricating film as a function of the relaxation rate.

$$v^1(0, y^2) = v^1(2\pi, y^2)\,, \quad v^2(0, y^2) = v^2(2\pi, y^2)\,. \tag{11}$$

For the density, we also specified the periodic boundary conditions along the coordinate y^1:

$$\rho(0, y^2) = \rho(2\pi, y^2)\,. \tag{12}$$

Problem (1)–(12) was solved numerically for different values of the internal relaxation time and a fixed value of the geometric parameter of $\lambda = 0.9$. In all the variants, we observed the process of establishment of the steady-state field of velocity and pressure. The calculations were carried out with the use of a finite-difference grid containing 120 nodes along the coordinate y^1 and 16 nodes along the coordinate y^2. We prescribed the fields of velocity $v^1 = U/h(\varphi)$ and $v^2 = 0$ and constant density ρ_0 as the initial conditions. The value of ρ_0 was determined by selection, from the condition that the minimum pressure in the fluid be 1 atm upon the establishment of the steady-state regime of flow.

Since the present work sought to investigate the influence of relaxation on the characteristics of a lubricating film, the viscosity coefficients η and μ were set constant for the sake of simplicity. The η/μ ratio varied from 1 to 100, but we failed to reveal the influence of the volume viscosity on the steady-state flow.

According to the results of the numerical modeling, we analyzed the characteristics of the steady-state flow for each of the variants considered. In particular, we investigated the pressure profile on the shaft surface and the carrying capacity of the lubricating film $W = R_{sh}\int_0^{2\pi} F(\varphi)d\varphi$. These quantities are represented in dimensionless form in Figs. 2 and 3. Figure 4 shows the distribution of the longitudinal velocity hv^1 for two variants with different relaxation times and $U = 10$ m/sec. For convenience of comparison to the results obtained earlier, transition from the dimensional quantities to dimensionless ones was carried out in accordance with the relations [4]

$$p = 12U\mu h_0^{-3/2}\gamma^{-1/2}\bar{p}\,, \quad W = 12U\mu h_0^{-1}\gamma^{-1}\overline{W}\,, \tag{13}$$

where $h_0 = \Delta(1 - \lambda)$ and $\gamma = \Delta(2R_{sh}R_r)^{-1}$. The relative relaxation rate was characterized by the dimensionless parameter $s = \tau_0/\tau_1$, $\tau_0 = h_0^{1/2}\gamma^{-1/2}U^{-1}$.

It is noteworthy that the parameter $q = (Uh_0)^{-1}Q$ significantly differed from unity: $|q - 1| \sim 0.35$. This rules out the application of the method of reduction to the one-dimensional equation used in [4]. We recall that the parameter q in [4] differed only slightly from unity: $|q - 1| \sim 0.1$.

The relative relaxation rate in the calculated variants (parameter s) varied from a value of 2.2 (the slowest relaxation) to values of instantaneous, in practice, relaxation of $s > 20$. Values lower than 2.2 require that the numerical scheme used be substantially complicated; therefore, we did not consider them. The case $s < 2.2$ is peculiar due to the

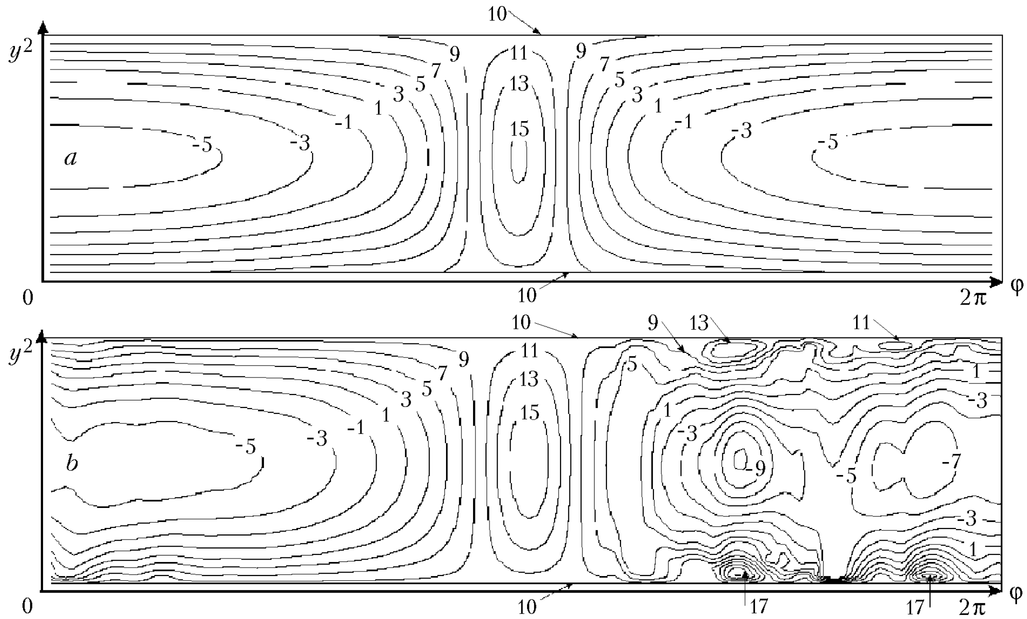


Fig. 4. Isolines of the longitudinal velocity component hv^{-1} for the cases $s > 20$ (a) and $s = 2.2$ (b).

fact that in the relaxation time corresponding to such values, the particles of the medium are able to cover a distance exceeding the longitudinal dimension of the region of flow.

From an analysis of the data in Figs. 2–4 it follows that, as the internal relaxation time increases, the pressure decreases throughout the length of the gap and the carrying capacity of the lubricating film decreases. These results qualitatively correspond to those obtained earlier in [4]. For $s > 20$, the relaxation effects become insignificant.

In slow relaxation, we observe a fairly complex flow structure with vortices that disappears when $s > 20$ (see Fig. 4). Thus, for the viscoelastic lubricant we have differences from the Reynolds theory in both the qualitative flow pattern and the quantitative characteristics.

NOTATION

A_1 and A_2, dimensionless weight factors characterizing the contribution of internal relaxation processes; $A(\tau)$, weight factor characterizing the contribution of internal relaxation processes, sec^{-1}; b_{α}^{a}, matrix of transition from the curvilinear coordinate system to a Cartesian one; $e^{\alpha\beta}$, contravariant components of the strain-rate tensor; e_{d}^{ab}, deviation component of the strain-rate tensor, sec^{-1}; e_{s}, spherical component of the strain-rate tensor, sec^{-1}; E, modulus of dilatation for the lubricant, Pa; $F(\varphi)$, vertical component of the force acting on the shaft surface from the lubricating film, kg/(m·sec^2); g, determinant of the metric tensor; $g_{\alpha\beta}$, covariant components of the metric tensor; $h(\varphi)$, width of the gap between the shaft and the ring, m; $h_{\mathrm{sh}}(\varphi)$, shaft-surface equation, m; $h_{\mathrm{r}}(\varphi)$, ring-surface equation, m; h_0, width of the gap between the shaft and the ring in the narrowest part of the channel, m; $K(t)$, relaxation kernel, sec^{-1}; p, pressure, Pa; p_*, atmospheric pressure, Pa; $\bar{p}$, dimensionless pressure; q, dimensionless flow rate; Q, rate of flow through the cross section of the gap for steady-state flow, m^2/sec; R_{sh}, radius of the shaft, m; R_{r}, radius of the ring, m; s, dimensionless relaxation parameter; t, time, sec; t_0, auxiliary variable of time integration, sec; U, linear rotational velocity of the shaft and the ring, m/sec; W, carrying capacity of the lubricating film per unit length, kg/sec^2; $\overline{W}$, dimensionless carrying capacity; v^{α}, contravariant components of the velocity vector; x^{a}, Cartesian coordinates, m; y^{α}, curvilinear coordinates; γ, parameter making a quantity dimensionless, m^{-1}; Δ, difference of the radii of the ring and the shaft, m; ε, eccentricity, m; η, volume coefficient of viscosity, Pa·sec; λ, dimensionless geometric parameter; μ, coefficient of shear viscosity, Pa·sec; $\pi^{\alpha\beta}$, momentum flow tensor; ρ, density of the lubricant, kg/m^3; ρ_0, initial value of the density, kg/m^3; ρ_*, fixed value of the density, corresponding to atmospheric pressure, kg/m^3; τ_0, characteristic time of traversal of the region of large pressure gradients by a particle of a medium, sec; τ_1, characteristic relaxation time, sec; τ_2 and τ_3, auxiliary parameters with the dimension of time, sec; $\tau^{\alpha\beta}$, contravariant components of the viscous stress tensor; τ_d^{ab}, deviation component of the viscous stress tensor, Pa; τ_s, spherical component of the viscous stress tensor, Pa; φ, angular coordinate; ω_{sh}, angular rotational velocity of the shaft, sec^{-1}; ω_r, angular rotational velocity of the ring, sec^{-1}. Subscripts and superscripts: *a* and *b* correspond to the Cartesian coordinates; sh, shaft; r, ring; d, deviation-tensor component; s, spherical-tensor component; α and β correspond to the curvilinear coordinates.

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